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Stable Subcritical Solutions for a Class of Variational Problems

GEORGE H. KNIGHTLY*

Department of Mathematics and Statistics, University of Massachusetts, Amherst, Massachusetts 01002

AND

D. SATHER†

Department of Mathematics, University of Colorado, Boulder, Colorado 80302

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1. INTRODUCTION

It is well-known that a number of important physical problems lead to the study of a nonlinear equation of the form

$$G(v, \sigma) = 0, \qquad v \in \mathscr{B}, \quad \sigma \in \mathbb{R}^1, \tag{1.1}$$

where $\mathscr{B}$ is a real Banach space, and $G: \mathscr{B} \times \mathbb{R}^1 \to \mathscr{B}$ is a smooth mapping with $G(0, \sigma) = 0$, $\sigma \in \mathbb{R}^1$. In many such problems the trivial solution is stable for σ less than some critical value σ_c and unstable for $\sigma > \sigma_c$. (In this paper, stability means "linearized stability"; that is, the stability or instability of a solution (v^*, σ^*) of (1.1) is determined by the spectrum of the derived operator $\partial G/\partial v$ at (v^*, σ^*).) If a nontrivial solution of (1.1) exists for $\sigma < \sigma_c$ it is said to be a subcritical solution. The existence or nonexistence of "large" stable, subcritical solutions is perhaps one of the major unsolved problems in physical situations described by equations of type (1.1). Although this problem is very difficult in general it is sometimes relatively easy to determine the stability properties of a subcritical solution in special situations. For example, a subcritical branch (v, σ), $\sigma < \sigma_c$, of solutions of (1.1) that bifurcates from the trivial solution at $\sigma = \sigma_c$ is usually unstable for (v, σ) near $(0, \sigma_c)$ (see [13–15]). A "typical" bifurcation diagram involving subcritical branching from $(0, \sigma_c)$ is shown in Fig. 1. The branches in Fig. 1

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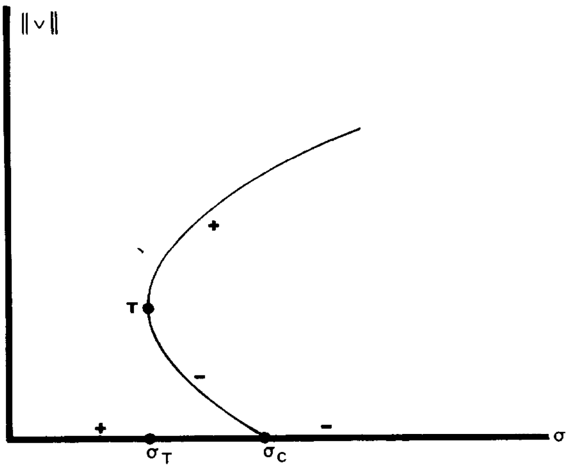

FIGURE 1

identified by a + sign are stable whereas the branches identified by a − sign are unstable. It is usually assumed that if a subcritical branch "turns around" at a point such as T in Fig. 1 then the branch gains stability and remains bounded for σ near σ_T. Such behavior is often used as a theoretical explanation of snap-buckling in shells, hysteresis-type effects in reaction–diffusion systems, etc. (e.g., see [8, 15]). If the null space of the linearized operator for (1.1) at T is "one-dimensional" in some sense, it is possible to give conditions under which the subcritical branch either bifurcates at T (e.g., see [1, 17]) or turns around and actually gains stability at T (e.g., see [14, 17]). On the other hand, since in most nonlinear problems the subcritical branch being considered is not known explicitly, results that depend upon knowledge of the solution at a critical point such as T are not always satisfactory. Even less satisfactory is the fact that if the dimension of the null space of the linearized operator for (1.1) at T is not one-dimensional, then a subcritical branch of (1.1) may or may not gain stability when it turns around. For example, let $\mathscr{B} = \mathbb{R}^2$ and consider the gradient system

$$
\begin{aligned}
0 &= w_1 - \sigma w_1 + \alpha^2 w_1 - 2\alpha w_1^2 + w_1^3 - w_1 w_2^2, \\
0 &= w_2 - \sigma w_2 + \alpha^2 w_2 + 2\alpha w_2^2 + w_2^3 - w_1^2 w_2.
\end{aligned}
\tag{1.2}
$$

It is easy then to see that the solution curve $\mathscr{C} = \{(w, \sigma)\colon w_2 = 0 \text{ and } \sigma = 1 + (w_1 - \alpha)^2\}$ fails to gain stability when it turns around at $(w_1, w_2, \sigma) = (\alpha, 0, 1)$; the eigenvalues of the linearized problem along $\mathscr{C}$ are $\pm 2w_1(w_1 - \alpha)$ so that both eigenvalues change sign at $(\alpha, 0, 1)$. Further analysis shows, in fact, that there are no stable subcritical solutions ("large" or "small") of (1.2).

Such an example shows clearly that some additional structure must be present in the problem being studied in order to guarantee that subcritical branches gain stability when they turn around. One way to introduce additional structure into the problem is to assume that a selection principle is operating so that only stable rather than unstable solutions of (1.1) are chosen as possible physical states of the problem. The present paper is an attempt to model such phenomena by formulating a selection principle that relates the selection of stable states to the structure of the governing equations rather than to the properties of the unknown solutions. In this way one may be able to construct part of the stable "upper branch" directly without knowing the properties of the subcritical branch either "below" the turning point T or at the point T itself. For certain problems in which a selection principle plays a role, such an approach would seem to model the physical process more closely than an approach that involves only an analysis of Eq. (1.1).

In order to illustrate the basic ideas we consider a class of variational problems suggested by some buckling problems in the nonlinear theory of shells (see [5–7, 13]). Such problems can be reduced to the study of what we shall call a state equation having the form

$$F(w, \lambda, \alpha) \equiv w - \lambda A w + \alpha^2 A^2 w + \alpha Q(w) + C(w) = 0,$$
$$w \in \mathscr{H}, \qquad \lambda \in \mathbb{R}^1, \qquad \alpha \in \mathbb{R}^1_+. \tag{$*$}$$

Here $\mathscr{H}$ is a real Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$, λ is some sort of "load" parameter, and α is some sort of "structural" or "geometric" parameter. The linear operator $A: \mathscr{H} \to \mathscr{H}$ is compact, selfadjoint and positive. The nonlinear operators $Q: \mathscr{H} \to \mathscr{H}$ and $C: \mathscr{H} \to \mathscr{H}$ are continuous, homogeneous, polynomial operators of degree 2 and 3, respectively, and are gradients of the functionals

$$q(u) \equiv \tfrac{1}{3}(Q(u), u) \qquad \text{and} \qquad c(u) \equiv \tfrac{1}{4}(C(u), u), \qquad u \in \mathscr{H}. \tag{1.3}$$

Some specific applications leading to equations of the form $(*)$ with $\alpha \ll 1$ are described in [5, 6].

The linear eigenvalue problem associated with $(*)$ is

$$L_\lambda w \equiv w - \lambda A w + \alpha^2 A^2 w = 0, \qquad w \in \mathscr{H}, \quad w \neq 0. \tag{1.4}$$

If we denote the smallest (positive) eigenvalue of (1.4) by $\lambda_c \equiv \lambda_c(\alpha)$, then the trivial solution, $w = 0$, of $(*)$ is stable for $\lambda < \lambda_c$ and unstable for $\lambda > \lambda_c$. In the applications described in [5–7] the smallest eigenvalue $\lambda_c = \lambda_c(\alpha)$ is the "classical buckling load" of the shell.

Since the operators in Eq. $(*)$ are gradient operators, a selection principle can be introduced into the problem in the following way. Let $E: \mathscr{H} \times \mathbb{R}^2 \to \mathbb{R}^1$ be the "energy" functional given by

$$E(w, \lambda, \alpha) \equiv (w, w) - \lambda(Aw, w) + \alpha^2(Aw, Aw) + \frac{2\alpha}{3}(Q(w), w) + \frac{1}{2}(C(w), w). \tag{1.5}$$

(In the applications described in [5–7] the relevant functional E of the form (1.5) is, in fact, a measure of the potential energy of a buckled state of the shell.) Then it is easy to see that $E = 0$ for the trivial solution, $w = 0$, of $(*)$ and, for fixed λ and α, $\operatorname{grad} E = 2F$. Therefore, if $\mathscr{C}$ is a smooth solution curve of $(*)$, parameterized by τ, then along $\mathscr{C}$

$$\frac{dE}{d\tau} = -\frac{d\lambda}{d\tau}(Aw, w). \tag{1.6}$$

Since A is positive, E decreases (increases) along such curves $\mathscr{C}$ as λ increases (decreases).

We assume in the following discussion of Fig. 2 that α is fixed. If a "typical" subcritical branch $\mathscr{C}$ such as that shown in Fig. 2 turns around and gains stability, we have the following situation: as (w, λ) moves along $\mathscr{C}$ from $(0, \lambda_c)$ through T, E is positive and increasing on the "lower branch" below T and decreasing on the "upper branch." Thus, it is natural to

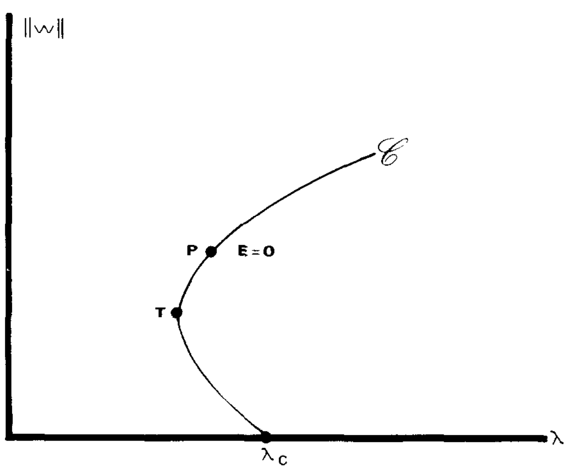

FIGURE 2

conjecture that what we shall call a "selection principle" is related to the existence, on the upper branch, of a point P at which $E=0$ is a local minimum of E for fixed λ (and α). In particular, such a selection principle should in some sense guarantee that P lies on an upper branch. Note that a point P at which $E=0$ cannot coincide with the turning point T of a subcritical branch such as that shown in Fig. 2. It follows for E sufficiently small that there should also be points on the upper branch that correspond to stable solutions of $(*)$ with $E \neq 0$.

The basic idea of the paper can now be stated as follows. Instead of solving only the state equation $(*)$ in $\mathscr{H}$ for "small" solutions $w = w(\lambda, \alpha)$ as is often done, we solve the system of equations

$$\begin{aligned} &F(w, \lambda, \alpha) = 0, \\ &E(w, \lambda, \alpha) = e\alpha^4, \qquad w \in \mathscr{H}, \quad \lambda \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1, \quad e \in \mathbb{R}^1, \end{aligned} \qquad (**)$$

in $\mathscr{H} \times \mathbb{R}^1$ for solutions $w = w(\alpha, e)$, $\lambda = \lambda(\alpha, e)$ with $\lambda(\alpha, e) < \lambda_c(\alpha)$, when (α, e) is near $(0, 0)$. Then, *for fixed* $\alpha = \alpha_0$ *near* $\alpha = 0$, a subcritical branch $(w(\alpha_0, e), \lambda(\alpha_0, e))$, $|e| < e_0$, is a valid candidate for a stable solution branch of the state equation $(*)$ for $|e| < e_0$ because one expects the condition

$$E(w(\alpha_0, e_0), \lambda(\alpha_0, e_0), \alpha_0) = e_0 \alpha_0^4, \qquad |e| < e_0$$

to imply that $(w(\alpha_0, e), \lambda(\alpha_0, e))$, $|e| < e_0$, is part of an upper branch. In the present paper we show that under certain natural conditions on Q and C the above approach always yields stable, subcritical solution branches of $(*)$ for fixed $\alpha = \alpha_0$ and $|e| < e_0$, when α_0 and e_0 are sufficiently small. Such solutions may even be considered as "large" solutions of $(*)$ because they are both subcritical and stable whereas "small" subcritical solutions branching from $w = 0$ at $\lambda = \lambda_c$ are always unstable.

Although the nonlinear problems considered here are variational it is expected that the main ideas of the paper are applicable to a wide range of physical phenomena governed by selection principles. For example, the main ideas of the paper also apply to certain Bénard-type convection problems in fluid mechanics as well as to certain kinds of reaction–diffusion problems for chemical systems. Such problems are not variational but they lead to a system of "reduced selection equations" that is of the same type as the finite-dimensional variational equations given in (2.8) below. The physical interpretation of the condition "$E = 0$" and the associated selection principle would, of course, be different for such problems and would not be related to a potential energy functional.

An application of some of the results of this paper to the buckling of a cylindrical panel is outlined in [8]. In such a problem it appears likely for small α that the value of λ, at a point (w, λ) at which $E = 0$, is an "intermediate buckling load" of the panel (see, e.g., [2; 9, Sect. 9; 11] for discussions of the concepts of "minimum buckling loads" and "intermediate buckling loads" in elastic shell theory). Thus, in addition to the stability of w and the subcritical nature of λ, the actual value of λ may also be of interest at a point (w, λ) at which $E = 0$.

2. The Reduced Selection Equations

In this section we reduce the problem of solving system $(**)$ in $\mathscr{H} \times \mathbb{R}^1$ to a finite-dimensional problem and derive what we shall call the *reduced selection equations*; we shall see that solving the reduced selection equations leads to solutions of $(*)$ that are both subcritical and stable. The splitting methods used to accomplish this reduction are closely related to the Lyapunov–Schmidt method in bifurcation theory.

We begin by describing briefly the linear theory for Eq. $(*)$. We shall say that λ is an eigenvalue of the linear, selfadjoint operator L_λ defined by (1.5) if $L_\lambda u = 0$ for some $u \in \mathscr{H}$, $u \neq 0$. The eigenvalues of L_λ are related to the characteristic values of A in the following way. Since

$$L_{\lambda_0} = (I - \mu^+ A)(I - \mu^- A) = (I - \mu^- A)(I - \mu^+ A), \tag{2.1}$$

where $\mu^\pm = (1/2)[\lambda_0 \pm (\lambda_0^2 - 4\alpha^2)^{1/2}]$, it follows that λ_0 is an eigenvalue of L_λ if and only if one of the numbers μ^+ or μ^- is a characteristic value of A. Therefore, λ_0 is an eigenvalue of L_λ if and only if $\lambda_0 = \mu_0 + \alpha^2 \mu_0^{-1}$ for some characteristic value μ_0 of A. In particular, if μ_1 denotes the smallest (positive) characteristic value of A then, for α sufficiently small,

$$\lambda_c = \mu_1 + \alpha^2 \mu_1^{-1} \tag{2.2}$$

determines the smallest (positive) eigenvalue of L_λ. We always assume α is so small that λ_c in (2.2) is the smallest eigenvalue of L_λ.

Let $\mathscr{M}$ denote the null space of $I - \mu_1 A$. Note that, by assumption, $\dim \mathscr{M} = m < \infty$. Then $\mathscr{H}$ can be decomposed as the direct sum $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^\perp$, where $\mathscr{M}^\perp$ is the orthogonal complement of $\mathscr{M}$ in $\mathscr{H}$. If S denotes the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$ then an element $w \in \mathscr{H}$ can be written as $w = z + Z$, where $z = Sw \in \mathscr{M}$ and $Z = (I - S)\, w \in \mathscr{M}^\perp$. Note that A maps $\mathscr{M}$ into $\mathscr{M}$ so that $AS = SA$.

We seek solutions of system $(**)$ that have the form

$$w = \alpha(u + \alpha U), \qquad u \in \mathscr{M}, \quad U \in \mathscr{M}^\perp, \tag{2.3a}$$

$$\begin{aligned} \lambda &= \mu_1 + \alpha^2 \mu_1^{-1} + \mu_1 \alpha^2 t \\ &= \lambda_c + \mu_1 \alpha^2 t, \qquad t \in \mathbb{R}^1. \end{aligned} \tag{2.3b}$$

We shall see that the form chosen for w in (2.3a) is a natural one that balances the contributions of the quadratic and cubic operators in $(*)$; in particular, the scaling in (2.3a) provides the possibility of obtaining "large" solutions of $(*)$. The form of λ in (2.3b) is essentially determined by the fact that we want to construct subcritical solutions of $(*)$. Note that if (w, λ) is a solution of $(*)$ for fixed α, and if (w, λ) is of the form (2.3) with $t < 0$, then $\lambda < \lambda_c$ so that (w, λ) is a subcritical solution of $(*)$. Thus, in what follows we seek solutions of system $(**)$ of the form (2.3) with $t < 0$.

If we substitute (2.3) into $(*)$, project the resultant equation onto $\mathscr{M}^{\perp}$ and $\mathscr{M}$ with $I - S$ and S, respectively, and cancel the factors α^2 and α^3 in the respective equations, we obtain the following equations in $\mathscr{M}^{\perp}$ and $\mathscr{M}$:

$$\begin{aligned} 0 &= U - \mu_1 AU + \alpha(I - S)[Q(u + \alpha U) + C(u + \alpha U)] \\ &\quad + \alpha^2 [A^2 U - (\mu_1^{-1} + \mu_1 t) AU], \qquad (2.4) \\ 0 &= -tu + SQ(u + \alpha U) + SC(u + \alpha U). \qquad (2.5) \end{aligned}$$

Substituting (2.3) into the functional E in (1.5), setting $E = e\alpha^4$, $e \in \mathbb{R}^1$, and cancelling the factor α^4, we obtain a third equation in $\mathbb{R}^1$, namely

$$\begin{aligned} 0 &= -t \|u\|^2 + \tfrac{2}{3}(Q(u + \alpha U), u + \alpha U) + \tfrac{1}{2}(C(u + \alpha U), u + \alpha U) \\ &\quad + \|U\|^2 - \mu_1(AU, U) + \alpha^2 [(A^2 U, U) - (\mu_1^{-1} + \mu_1 t)(AU, U)] - e. \qquad (2.6) \end{aligned}$$

In order to solve (2.4) through (2.6) near $\alpha = e = 0$ we define $\mathscr{F}$: $\mathscr{M}^{\perp} \times \mathscr{M} \times \mathbb{R}^3 \to \mathscr{M}^{\perp} \times \mathscr{M} \times \mathbb{R}^1$ to be the mapping given by the right hand sides of (2.4), (2.5) and (2.6). Then $\mathscr{F}$ is a continuous polynomial operator and the Fréchet derivative of $\mathscr{F}$ with respect to (U, u, t) evaluated at $(U, u, t, \alpha, e) = (0, u^*, t^*, 0, 0)$ is given by

$$\begin{bmatrix} I - \mu_1 A & 0 & 0 \\ 0 & -t^* I + SQ'(u^*) + SC'(u^*) & -u^* \\ 0 & 2(-t^* u^* + SQ(u^*) + SC(u^*), \cdot) & -\|u^*\|^2 \end{bmatrix}. \qquad (2.7)$$

Since the operator $I - \mu_1 A$ has a bounded inverse on $\mathscr{M}^{\perp}$, it follows that the invertibility of (2.7) essentially reduces to the question of whether or not the restriction of the Fréchet derivative of $\mathscr{F}$ is invertible on $\mathscr{M} \times \mathbb{R}^1$. If we now formally set $U = 0$ and $\alpha = e = 0$ in (2.5) and (2.6), we obtain the so-called *reduced selection equations* associated with system $(**)$, namely

$$0 = -tu + SQ(u) + SC(u), \qquad u \in \mathscr{M}, \quad t \in \mathbb{R}^1, \qquad (2.8a)$$

$$0 = -t \|u\|^2 + \tfrac{2}{3}(Q(u), u) + \tfrac{1}{2}(C(u), u). \qquad (2.8b)$$

Clearly, if $(u^*, t^*) \in \mathscr{M} \times \mathbb{R}^1$ is a solution of system (2.8) with $u^* \neq 0$, and if the operator

$$\begin{bmatrix} -t^*I + SQ'(u^*) + SC'(u^*) & -u^* \\ 0 & -\|u^*\|^2 \end{bmatrix} \tag{2.9}$$

has a bounded inverse on $\mathscr{M} \times \mathbb{R}^1$, then the Fréchet derivative of $\mathscr{F}$ in (2.7) has a bounded inverse at $(0, u^*, t^*, 0, 0)$. Thus, by the implicit function theorem there exist analytic functions $U(\alpha, e)$, $u(\alpha, e)$ and $t(\alpha, e)$ such that $(U(\alpha, e),\ u(\alpha, e),\ t(\alpha, e),\ \alpha, e)$ is a solution of $\mathscr{F}(U, u, t, \alpha, e) = 0$ for $0 \leqslant \alpha < \alpha_1$, $|e| < e_1$. It then follows that the pair $(w^*(\alpha, e), \lambda^*(\alpha, e))$ defined by (2.3) is an analytic solution of system $(**)$ for $0 \leqslant \alpha < \alpha_1$, $|e| < e_1$. Thus, the problem of finding nontrivial solutions of system $(**)$ for (w, λ, α, e) near $(0, \mu_1, 0, 0)$ reduces to the problem of finding nontrivial solutions (u^*, t^*) of the reduced selection equations for which (2.9) has a bounded inverse on $\mathscr{M} \times \mathbb{R}^1$.

Remark 1. The reduced equation in (2.8a) is *not* the reduced bifurcation equation associated with Eq. $(*)$ at $\lambda = \lambda_c$ (e.g., see [6, 7, 13, 15]). Note that both Q and C appear in (2.8a) because of the form of the scaling in (2.3a) whereas typically only the lower order operator Q appears in the reduced bifurcation equation. An equation of the form (2.8a) also arises naturally in the treatment of Eq. $(*)$ by means of singularity theory; however, the derivation and interpretation of such an equation is quite different there (see, in particular, the discussion in [3, 10]).

Remark 2. Suppose that (u^*, t^*) is a solution of (2.8) with $u^* \neq 0$. Taking the inner product of (2.8a) with u^* and using (2.8b) to eliminate the term $(Q(u^*), u^*)$, one sees that t^* necessarily satisfies

$$t^* = -\frac{1}{2} \frac{(C(u^*), u^*)}{\|u^*\|^2}. \tag{2.10}$$

Thus, if $(C(u^*), u^*) > 0$ then $t^* < 0$ and the resultant solution $(w^*(\alpha, e), \lambda^*(\alpha, e))$ of system $(**)$ provides a subcritical solution branch $(w^*(\alpha_0, e), \lambda^*(\alpha_0, e))$ of $(*)$ of the form (2.3) for each *fixed* $\alpha = \alpha_0$, $0 < \alpha_0 < \alpha_1$, and e sufficiently small. In this sense a condition such as $(C(u), u) > 0$ for $u \in \mathscr{M}$, $u \neq 0$, is "necessary" for the existence of subcritical solutions of $(*)$ of the form (2.3).

3. Subcritical Solutions of System $(**)$

In this section we obtain nontrivial solutions of the reduced selection equations in (2.8) under certain natural assumptions on Q and C. These solutions of (2.8), in turn, generate subcritical solutions of system $(**)$ by means of the implicit function theorem. The stability of the resultant subcritical solution is considered in Section 4.

We have the following result when $\dim \mathcal{M} = 1$.

THEOREM 1. *Suppose that* $\dim \mathcal{M} = 1$ *and that* $\mathcal{M}$ *is spanned by* u_1. *Suppose further that*

$$a = (C(u_1), u_1) > 0 \qquad \text{and} \qquad b = (Q(u_1, u_1) \neq 0. \tag{3.1}$$

Then there exist $\alpha_1 > 0$ *and* $e_1 > 0$ *such that whenever* $0 \leqslant \alpha < \alpha_1$ *and* $|e| < e_1$ *system* $(**)$ *has a solution* $(w^*(\alpha, e), \lambda^*(\alpha, e))$ *of the form* (2.3) *with* $t = t(\alpha, e) < 0$ *and* $(w^*(\alpha, e), \lambda^*(\alpha, e)) \to (0, \mu_1)$ *as* $(\alpha, e) \to (0, 0)$, $\alpha > 0$.

Proof. If $u \in \mathcal{M}$ then $u = ru_1$, $r \in \mathbb{R}^1$. Taking the inner product of (2.8a) with u_1 and cancelling terms r and r^2, we see that the reduced selection equations in (2.8) when $\dim \mathcal{M} = 1$ are equivalent to the following system in $\mathbb{R}^2$:

$$0 = -t + br + ar^2, \tag{3.2a}$$

$$0 = -t + \tfrac{2}{3}br + \tfrac{1}{2}ar^2. \tag{3.2b}$$

These equations have the unique nontrivial solution

$$(r^*, t^*) = (-2b/3a, -2b^2/9a) \tag{3.3}$$

at which the Jacobian J satisfies $J = b/3 \neq 0$. The proof of Theorem 1 now follows from the implicit function theorem and the discussion at the end of Section 2.

In the general case it is convenient to obtain nontrivial solutions of the reduced selection equations by means of a variational argument. Consider the extremum problem

$$\inf_{u \in \mathcal{M}} \{f(u)\}, \tag{3.4}$$

where

$$\begin{aligned} f(u) &= \frac{q(u) + c(u)}{\|u\|^2} \qquad && \text{if} \quad u \neq 0 \\ &= 0 && \text{if} \quad u = 0. \end{aligned} \tag{3.5}$$

Here the functionals $q(u)$ and $c(u)$ are defined as in (1.3). We assume that Q and C satisfy the following hypotheses *on* $\mathcal{M}$:

(HQ) $\qquad (Q(u), u) \not\equiv 0, \qquad u \in \mathcal{M},$

(HC) $\qquad (C(u), u) > 0, \qquad u \in \mathcal{M} \quad \text{and} \quad u \neq 0.$

(Note that the conditions in (3.1) are equivalent to (HQ) and (HC) when $\dim \mathscr{M} = 1$.) Since $\mathscr{M}$ is finite-dimensional, hypotheses (HQ) and (HC) imply that the closely related extremum problem

$$\max_{v \in \mathscr{M}, \|v\| = 1} \left\{ \frac{q^2(v)}{4c(v)} \right\} \tag{3.6}$$

is well-defined. Thus, there exists an element $v_0 \in \mathscr{M}$, $\|v_0\| = 1$, such that $q_0^2/4c_0$ is the maximum value in (3.6) with $q_0 = (Q(v_0), v_0)$, $c_0 = (C(v_0), v_0)$, and $0 < q_0^2/4c_0 < \infty$. If $u \in \mathscr{M}$ and $u \neq 0$, we set $u = \rho v$, where $\|v\| = 1$. Then

$$f(u) = c(v)[\rho + q(v)/2c(v)]^2 - q^2(v)/4c(v) \geqslant -q_0^2/4c_0 \tag{3.7}$$

for all $u \in \mathscr{M}$, $u \neq 0$. Since $f(u^*) = -q_0^2/4c_0$ when $u^* = -(q_0/2c_0)\, v_0$, it follows that f has a finite negative minimum on $\mathscr{M}$ at $u^* \neq 0$. Therefore,

$$f'(u^*) = \frac{SQ(u^*) + SC(u^*)}{\|u^*\|^2} - \frac{2f(u^*)}{\|u^*\|^2}\, u^* = 0 \tag{3.8}$$

and

$$\begin{aligned}(f''(u^*)\, h, h) &= \frac{(Q'(u^*)\, h, h) + (C'(u^*)\, h, h)}{\|u^*\|^2} - \frac{2f(u^*)\, \|h\|^2}{\|u^*\|^2} \\ &\geqslant 0, \qquad h \in \mathscr{M}. \end{aligned} \tag{3.9}$$

If we now set $t^* = 2f(u^*)$, then (3.8) implies that (u^*, t^*) is a nontrivial solution of (2.8a). Moreover, since we can rewrite (2.8b) as

$$\begin{aligned} 0 &= -t\, \|u\|^2 + 2q(u) + 2c(u) \\ &= (-t + 2f(u))\, \|u\|^2, \end{aligned} \tag{3.10}$$

it follows that (u^*, t^*) is also a solution of (2.8b). We have established the following result when $\dim \mathscr{M} \geqslant 2$.

THEOREM 2. *Suppose that* $\dim \mathscr{M} = m \geqslant 2$, *and that* Q *and* C *satisfy hypotheses* (HQ) *and* (HC). *Suppose that* (u^*, t^*) *is a nontrivial solution of* (2.8) *determined by the variational problem* (3.4) *as in the above discussion, and suppose, in addition, that* 0 *is not an eigenvalue of the operator* $M: \mathscr{M} \to \mathscr{M}$ *defined by*

$$M = -t^* I + SQ'(u^*) + SC'(u^*). \tag{3.11}$$

Then there exists $\alpha_1 > 0$ *and* $e_1 > 0$ *such that whenever* $0 \leqslant \alpha < \alpha_1$ *and*

$|e| < e_1$ *system* $(**)$ *has a solution* $(w^*(\alpha, e), \lambda^*(\alpha, e))$ *of the form* (2.3) *with* $t = t(\alpha, e) < 0$ *and* $(w^*(\alpha, e), \lambda^*(\alpha, e)) \to (0, \mu_1)$ *as* $(\alpha, e) \to (0, 0)$, $\alpha > 0$.

The assumption in Theorem 2 that 0 is not an eigenvalue of M is a generic one and is the usual type of "Jacobian condition" required in higher dimensional variational problems such as (3.4). Under this assumption the operator in (2.9) has a bounded inverse on the finite-dimensional space $\mathscr{M} \times \mathbb{R}^1$ so that the proof of Theorem 2 follows by an application of the implicit function theorem as described in Section 2.

The results in Theorem 1 and Theorem 2 provide the existence of *subcritical* solutions of system $(**)$, under what are probably the most natural assumptions on Q and C. It remains, of course, to show that the above construction actually guarantees stability of the subcritical solutions $(w^*(\alpha, e), \lambda^*(\alpha, e))$ for α and e sufficiently small.

4. Stability of the Subcritical Solutions

In this section we show that the subcritical solutions of $(**)$ constructed in Section 3 are stable for α and e sufficiently small.

Let $(w^*, \lambda^*) = (w^*(\alpha, e), \lambda^*(\alpha, e))$ be the subcritical solution of $(**)$ constructed in Theorem 1 or Theorem 2. Then the derived operator $D = \partial F/\partial w$ for Eq. $(*)$ at (w^*, λ^*) is given by

$$D(w^*, \lambda^*, \alpha) = I - \lambda^* A + \alpha^2 A^2 + \alpha Q'(w^*) + C'(w^*). \tag{4.1}$$

Since the solution (w^*, λ^*) is of the form (2.3) for $0 \leqslant \alpha < \alpha_1$ and $|e| < e_1$, the derived operator can be written as

$$\begin{aligned} D(w^*, \lambda^*, \alpha) = I - \mu_1 A + \alpha^2 [A^2 - (\mu_1^{-1} + \mu_1 t^*) A + Q'(u^*) \\ + C'(u^*) + K(\alpha, e)], \end{aligned} \tag{4.2}$$

where (u^*, t^*) is the appropriate solution of the reduced selection equations in (2.8), and $K(\alpha, e)$ is a bounded operator such that $\|K(\alpha, e)\| \to 0$ as $(\alpha, e) \to (0, 0)$, $\alpha > 0$. Since, in (4.1), $Q'(w^*)$ and $C'(w^*)$ are symmetric operators (e.g., see [16, p. 56]), the operator $D \equiv D(w^*, \lambda^*, \alpha)$ is a symmetric perturbation of a selfadjoint operator and is therefore itself selfadjoint with only real eigenvalues [4, p. 287]. Moreover, since there exists $\gamma > 0$ such that

$$((I - \mu_1 A) v, v) \geqslant \gamma \|v\|^2 \tag{4.3}$$

for all $v \in \mathscr{M}^\perp$, the isolation distance $2d$ of the eigenvalue $\sigma = 0$ of $I - \mu_1 A$ is positive. Consequently, for α and e sufficiently small, the spectrum of D consists of a part containing the $m = \dim \mathscr{M}$ critical eigenvalues $\sigma_i = \sigma_i(\alpha, e)$

that lie in the interval $(-d, d)$, and a second part that lies in the interval (d, ∞) (e.g., see [4, p. 290ff.]). The definition of stability used in the present paper is as follows: If, *for fixed* α *and* e sufficiently small, the eigenvalues $\sigma_i(\alpha, e)$ of D are all positive then $w^*(\alpha, e)$ is stable at $\lambda = \lambda^*(\alpha, e)$; if at least one of the eigenvalues $\sigma_i(\alpha, e)$ is negative then $w^*(\alpha, e)$ is unstable at $\lambda = \lambda^*(\alpha, e)$. If neither of these cases hold the stability is indeterminate. Such a definition is consistent with the usual notions of "linearized stability." Note that according to the above definition of stability, for fixed $\alpha > 0$, the trivial solution $w = 0$ of $(*)$ is stable for $\lambda < \lambda_c(\alpha)$ and unstable for $\lambda > \lambda_c(\alpha)$.

The following result shows that the stability of the subcritical solutions of $(**)$ constructed in Section 3 by means of a selection principle is, in fact, a consequence of that construction.

THEOREM 3. *Under the assumptions of either Theorem* 1 *or Theorem* 2, *the resultant subcritical solution* $w^*(\alpha, e)$ *is a stable solution of Eq.* $(*)$ *at* $\lambda = \lambda^*(\alpha, e)$ *for* α *and* e *sufficiently small.*

Proof. Suppose that $\dim \mathscr{M} = m \geqslant 1$. Let $\{u_1, \ldots, u_m\}$ be a basis for $\mathscr{M}$ such that

$$(Au_i, u_j) = \delta_{ij} \qquad (i, j = 1, 2, \ldots, m). \tag{4.4}$$

The approach in [13, Sect. 4] can now be used to show that the problem of determining the signs of the critical egenvalues $\sigma_i(\alpha, e)$ of D for α and e sufficiently small is equivalent to the problem of determining the signs of the eigenvalues of a symmetric $m \times m$ matrix B whose elements B_{ij} are given by

$$B_{ij} = (Mu_i, u_j) \equiv ((-t^*I + SQ'(u^*) + SC'(u^*))\, u_i, u_j), \qquad i, j = 1, \ldots, m. \tag{4.5}$$

If $\dim \mathscr{M} = 1$ and if $u^* = r^* u_1$, where r^* is defined as in (3.3), then the sign of (Mu_1, u_1) in (4.5) is the same as the sign of

$$\begin{aligned}(Mu^*, u^*) &= -t^* \|u^*\|^2 + 2(Q(u^*), u^*) + 3(C(u^*), u^*)\\ &= \tfrac{1}{2}(C(u^*), u^*).\end{aligned} \tag{4.6}$$

Here we have also used (2.8b), (2.10) and the Euler identities $(Q'(u)\, u, u) = 2(Q(u), u)$ and $(C'(u)\, u, u) = 3(C(u), u)$ (e.g., see [12]). Since $(C(u^*), u^*) > 0$ by assumption (3.1) in Theorem 1, it follows that the subcritical solution $w^*(\alpha, e)$ of $(*)$ at $\lambda = \lambda^*(\alpha, e)$ is stable for α and e sufficiently small.

If $\dim \mathscr{M} = m \geqslant 2$ then (3.9) and the definition of M in (3.11) imply that the eigenvalues of the matrix B in (4.5) are always nonnegative. Since 0 is not an eigenvalue of M by assumption in Theorem 2, it follows that the critical eigenvalue $\sigma_i(\alpha, e)$ of D are all positive for α and e sufficiently small. Thus, under the assumptions of Theorem 2, the resultant subcritical solution $w^*(\alpha, e)$ of $(*)$ at $\lambda = \lambda^*(\alpha, e)$ is also stable for α and e sufficiently small.

The results of Section 3 and Section 4 show that under certain natural conditions on Q and C (and generic conditions on the solutions of the reduced selection equations) the approach described in this paper always generates stable subcritical solutions of $(*)$. Simple examples (e.g., see (1.2) in the Introduction) show that restrictions such as (HQ) and (HC) are "necessary" in the sense that, if they do not hold, then there are examples of the form $(*)$ for which there are no stable subcritical solutions.

Remark 3. Typically, one expects to generate a nontrivial, *subcritical* solution of $(*)$ with $E = 0$, from each negative isolated stationary value of the functional f in (3.5). In this way the methods of the paper may lead to the existence of *several* stable subcritical branches corresponding to negative isolated, relative, minimum values of f. The t^*-values associated with such stable solutions would in turn suggest a certain ordering of the stable subcritical branches (e.g., an ordering by values of E at a given value of λ). In specific applications, such an ordering of subcritical branches could be useful in explaining why a physical system selects a particular subcritical physical state.

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